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Are modified gravity models free of ghosts?

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Abstract

Recently, attention has been given to modified gravity models, as they may represent alternatives to quintessence models for an explanation to today's acceleration of the universe. We review the main results developed in De Felice, Hindmarsh and Trodden (2006 *J. Cosmol. Astropart. Phys.* JCAP08(2006)005 (*Preprint* astro-ph/0604154)) and Calcagni, de Carlos and De Felice (2006 *Nucl. Phys.* B (*Preprint* hep-th/0604201)), where we showed that those theories that involve a coupling between a scalar field with the Gauss–Bonnet curvature combination, in general possess both ghosts degrees of freedom and classical instabilities. We believe these constraints put severe bounds on these types of modifications of gravity. We add a critical assessment of the present understanding of these bounds.

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1. Introduction

According to the principles of general relativity (GR), the equations of motion for gravity should not include third or higher order derivatives in the metric, and they should be linear in the second derivatives of the metric elements. In four dimensions, the most general Lagrangian that satisfies these requirements is the Einstein–Hilbert action with the addition of a cosmological constant. This simple model, after adding both radiation and dark matter, can actually fit today's data. The only problem related to this classical picture is that the value of the cosmological constant, in this approach, is not predicted. In fact, the value necessary to fit today's data is well below the value that particle physics can predict.

As there are no satisfactory models at the moment that explain the tiny value of the cosmological constant, the attention has been shifted from solving the cosmological constant problem itself to finding a dynamical model, which can explain today's acceleration and the coincidence problem. In this sense, probably the quintessence models have been the most promising ones [1, 2]. However, because gravity is the least known among forces, some have argued that, since the acceleration of the universe at large scales may be interpreted as a deviation from standard GR, we need a search running for sensible modifications of the Einstein–Hilbert action. In fact, some of these new proposals required the introduction of extra dimensions [3], others instead, used general functions of the Ricci scalar [4]. These last ones, in particular, proved to be equivalent to a scalar–tensor theory [5]. In the CDDETT paper [6], the authors introduced a Lagrangian made of powers of curvature invariants of second order. This model will be the one discussed in this paper.

The paper is divided as follows. In section 2, the CDDETT model is shortly reviewed. The theoretical bounds about instabilities for these models are discussed in section 3. Finally, section 4 is devoted to the conclusions.

2. The actions considered

The action we study here is the one introduced in the CDDETT paper [6]. It can be written as follows:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}R - \theta_\mu \frac{\mu^{4n+2}}{(a_1R^2 + a_2R_{\alpha\beta}R^{\alpha\beta} + a_3R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})^n}\right], \tag{1}$$

where $\theta_\mu = \pm 1$, μ is a parameter with dimensions of mass, a_i are real numbers and n is an integer. Even though this Lagrangian has many free parameters, it was soon realized that if $a_2 \neq -4a_3$, a spin-2 ghost would be present. Essentially, this is related to the fact that the equations of motion for the metric become of fourth order. Therefore, there was some excitement thinking that choosing $a_2 = -4a_3$ would heal this class of Lagrangians from the presence of ghosts degrees of freedom. In this and in the remaining sections we will show that this is not the case: even though there are no more spin-2 ghosts, other degrees of freedom, both the scalar and tensor modes, may still become ghosts and unstable.

Choosing then $a_2 = -4a_3$ and $a_3 \neq 0$ (otherwise the theory would become an $f(R)$ scalar–tensor theory), the previous action can be rewritten as follows:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{\theta_\mu}{a_3^n}\frac{\mu^{4n+2}}{\left(bR^2 + R_{\mathrm{GB}}^2\right)^n}\right], \tag{2}$$

where $b \equiv a_1/a_3 - 1$ and $R_{\mathrm{GB}}^2 \equiv R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is the Gauss–Bonnet (GB) combination. In order to understand clearly the physical degrees of freedom, it is helpful to rewrite the previous Lagrangian in the following equivalent form:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left\{\frac{1}{2}[1 + 4b\sigma f(\phi)]R - U(\phi) - b\sigma^2 f(\phi) + f(\phi)R_{\mathrm{GB}}^2\right\}, \tag{3}$$

where

$$U(\phi) = \theta_\mu\mu^{4n+2}\frac{n+1}{\phi^n} \tag{4}$$

$$f(\phi) = a_3\theta_\mu\mu^{4n+2}\frac{n}{\phi^{n+1}}. \tag{5}$$

Therefore, it is clear that in this model there are two scalar fields of which one is non minimally coupled with R and the other with R_{GB}^2. This second coupling differentiates these models from scalar–tensor theories. In fact, in this case, a conformal transformation does not lead to an Einstein frame where, by definition, the scalars are minimally coupled with gravity. As we shall see later on, all the instabilities come from the coupling between ϕ and the Gauss–Bonnet combination. As the field ϕ in general is dynamical, the Gauss–Bonnet term does give a contribution to the 4D equations of motion. It is interesting to note that the equations of motion for all the fields are of second order. In fact, by calling $\chi = 2b\sigma f$, they can be written as follows:

$$\sigma = R, \tag{6}$$

$$\phi = a_3\big(bR^2 + R_{\mathrm{GB}}^2\big), \tag{7}$$

$$\begin{aligned}\left(\frac{1}{2}+\chi\right) R_{\alpha\beta} - \nabla_\alpha\nabla_\beta\chi + g_{\alpha\beta}\,\Box\,\chi - 2R\nabla_\alpha\nabla_\beta f + 2g_{\alpha\beta}R\,\Box\, f + 8R_{(\alpha\nu}\nabla_{\beta)}\nabla^\nu f \\ - 4R_{\alpha\beta}\,\Box\, f - 4g_{\alpha\beta}R^{\rho\sigma}\nabla_\rho\nabla_\sigma f - 4R_{(\alpha}{}^{\sigma\tau}{}_{\beta)}\nabla_\sigma\nabla_\tau f \\ - \frac{1}{2}g_{\alpha\beta}\left[\left(\frac{1}{2}+\chi\right)R - \frac{\chi^2}{4bf} - U\right] = 0,\end{aligned} \tag{8}$$

and the first two equations (found by varying the action with respect to σ and ϕ respectively) are second-order differential equations in the metric elements, whereas, from the third equation (found by varying the action with respect to the metric elements $g_{\mu\nu}$), it is clear that both the scalar fields are dynamical.

Therefore, these models have something in common with the Gauss–Bonnet cosmological models, whose action is written in terms of a *single* scalar field, which has a canonical kinetic term, and a coupling with the Gauss–Bonnet combination. More details about the study of ghosts and instabilities in the GB cosmological models can be found in [7].

3. Cosmological perturbations

When the CDDETT models were first introduced, the de Sitter solution was immediately found. However, the de Sitter background is unstable in these theories. Nevertheless, attractor solutions do exist and they are power-law solutions. Since there is a large parameter space that predicts accelerating power-law solutions, these models were proposed as alternatives to quintessence. After the first proposal of this model there were many studies of the instabilities and ghosts, but they were made assuming a de Sitter background [8–10]. The fact that this background is unstable suggests, in fact, that the instability/ghost analysis should be made first on the attractor power-law solutions. Furthermore, the fact that these instabilities need to be absent also in the past history of the universe suggests that this analysis should be actually done on a general FRW background.

Therefore, we will expand the action at second order in the fields about a general FRW background. We will consider then scalar, vector and tensor perturbations for the metric. After having chosen a background and broken Lorentz invariance, even though the field equations are still of second order, a field can indeed show an unstable behaviour. In particular, if we have a field for which the action is as follows:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}T(t)\dot{\phi}^2 - \frac{1}{2}S(t)\nabla\phi^2\right], \tag{9}$$

then the speed of propagation for this mode is $v = \sqrt{S/T}$. If S/T is negative then the speed of propagation becomes imaginary, and instabilities grow and the system becomes unstable. If $S/T > 1$ then there are superluminally propagating modes. Finally, if $T < 0$ we have a ghost degree of freedom, which, interacting with other fields, makes quantum field theory inconsistent.

As an example, considering the simple case $b = 0$, and only the tensor modes of the metric, $H^i{}_j$, which are transverse and traceless, it can be shown that expanding the action at second order in these fields one finds

$$\delta_g^2 S_{(2)}^T = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}H_{ij}\left(\frac{1}{2}+\frac{4}{a^2}(f''-\mathcal{H}f')\right)\triangle H^{ij}+\frac{1}{2}H_{ij}\left(\frac{1}{2}+\frac{4}{a^2}\mathcal{H}f'\right)\nabla_0\nabla^0 H^{ij}\right], \tag{10}$$

where a prime denotes differentiation with respect to the proper time η defined as $\mathrm{d}t = a\,\mathrm{d}\eta$, $\mathcal{H} \equiv a'/a$, and a is the scale factor for the FRW metric. Now it is clear that, since ϕ and $f(\phi)$ are constants on a de Sitter background, de Sitter is ghosts free. However, for a general FRW background this is not always the case. Furthermore, it is clear that the coupling that leads to this unstable behaviour is the coupling with the Gauss–Bonnet term, not the one with the Ricci scalar. Rewriting the factors in front of the second-order differential operators, one finds that

$$T(t) = 1 + 8H\dot{f} \tag{11}$$

$$S(t) = 1 + 8\ddot{f}. \tag{12}$$

The dynamics of ϕ (and consequently of $f(\phi)$) are not trivial, therefore, in order to know the behaviour of both S and T, one needs to solve the differential equations for the background.

For the general case $b \neq 0$ and the tensor modes, one can show that the following stability conditions need to be satisfied (see [11] for more details):

$$1 + 4bfR + 8H\dot{f} > 0, \tag{13}$$

$$0 < c_2^2 = \frac{1 + 4bfR + 8\ddot{f}}{1 + 4bfR + 8H\dot{f}} \leqslant 1. \tag{14}$$

In the case $b \neq 0$ both the Bardeen scalars propagate. The ghost condition for the scalar modes is equivalent to the previous ghost condition for the tensor modes, however from the expression of their speed, these extra two constraints can be written

$$0 < c_0^2 = 1 + \frac{32}{3Q_1}\dot{f}\dot{H} - \frac{8}{3Q_2}(\ddot{f} - \dot{f}H) \leqslant 1, \tag{15}$$

where

$$Q_1 = 4b(\dot{f}R + f\dot{R}) + 8\dot{f}H^2 \qquad \text{and} \qquad Q_2 = 1 + 4bfR + 8H\dot{f}. \tag{16}$$

In total one has five independent conditions for the background in order not to have instabilities, ghosts, and superluminal modes. A complete study of these conditions at all times is still a work in progress, but it is instructive to consider them to constrain the parameter space of the accelerating power-law solutions. The accelerating power-law attractors, for which $a(t) \propto t^p$, have exponent

$$p = \frac{2(n+1)(4n+1) - 3\alpha + \sqrt{9n^2\alpha^2 - 4(n+1)(4n+1)(5n+1)\alpha + 4(4n+1)^2(n+1)^2}}{4(n+1)}, \tag{17}$$

where $\alpha \equiv 6b/(6b+1)$.

A plot of the allowed values of α and n, after imposing the no-ghost constraints, is shown in figure 1.

It is important to note that the accelerating power-law solutions are attractors for the modified Friedmann equation. On the other hand, a large part of the parameter space of these attractors is excluded as the background becomes unstable because of the ghost propagating modes. Therefore, there is a clear difference between stability in the phase–space and stability of the metric perturbation modes, in particular the first one does not imply the second.

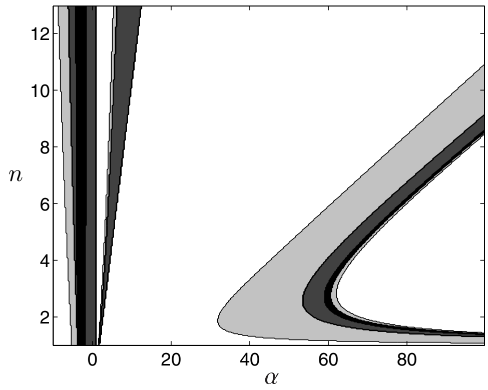

Figure 1. Contour plot in the (α, n) plane for the constraints. The light grey area corresponds to the region in which only the no-ghost constraint holds. The darker area represents the points at which both the no-ghost and the positive-squared-velocity conditions hold at the same time. Finally, the darkest region is the region of the plane at which all the constraints (no-ghost, $0 < c^2 < 1$) hold for both scalar and tensor modes.

4. Discussion

In classical mechanics, for a ghost on which only a conservative force acts, it is possible to write down the Lagrangian in the form $L = -T - U$, where $T = \frac{1}{2}m\vec{v}^2$. Therefore, a potential which would be stable for a non-ghost object is actually unstable for a ghost and vice versa. If one assumes to name V a normal-particle stable potential, then the equations of motion for a ghost coming from the Lagrangian $L_{\text{ghost}} = -T + V$ are identical of those for a standard particle. However, if $T = \frac{1}{2}m\dot{q}^2$ and $V = V(q)$, then the Hamiltonian (i.e. the total energy in this case) would be of opposite sign, as $H = -p^2/(2m) - V$, where $p = -m\dot{q}$. Therefore, in classical mechanics a ghost would be in general a particle with negative mechanical energy. If one considers a mechanical interaction between a ghost object and a normal one, then instabilities may actually arise. For example, for two harmonic oscillators, of which one is ghost-like, as in $L = -\frac{1}{2}\dot{q}_1^2 + \frac{1}{2}k_1q_1^2 + \frac{1}{2}\dot{q}_2^2 - \frac{1}{2}k_2q_2^2 + \lambda q_1q_2$, one finds that if $\lambda^2 > (k_1 - k_2)^2/4$ (which is always true if $k_1 = k_2$), then the variables q_i grow up exponentially. Otherwise, the orbits would be bound in a finite region of space.

In non-relativistic quantum mechanics, a ghost with Hamiltonian $\hat{H} = -\hat{p}_1^2\big/2m - \frac{1}{2}m\omega^2\hat{q}_1^2$ would have a negative spectrum $E_{n_1} = -\hbar\omega\big(n_1 + \frac{1}{2}\big)$. If we couple this ghost-like harmonic oscillator with a normal oscillator with the same mass and frequency through a time-dependent interaction, as in $V_{\text{int}} = \lambda\theta(T^2 - t^2)q_1q_2$ (where $\theta(x)$ is the Heaviside function), then the energy of the bound states at $t \to \pm\infty$ is given by $E_{1,2} = \hbar\omega(n_2 - n_1)$. It can be shown that the state, $(n_1 = 0, n_2 = 0)$ is indeed unstable. In fact, as far as first-order perturbation theory holds, after a time $2T$, the system decays into the isoenergetic state $(n_1 = 1, n_2 = 1)$ with a probability $P = (\lambda T/m\omega)^2$. In this case, the total energy of the system would not change but the ghost oscillator would give more and more energy to the normal oscillator. In fact, being the system initially in the state $(n_1 = 1, n_2 = 1)$ and switching on again the same perturbation, the probability for the system to go to the isoenergetic state $(n_1 = 2, n_2 = 2)$ is four times larger than the probability P to go to the other isoenergetic state $(n_1 = 0, n_2 = 0)$.

In quantum field theory, if one chooses the propagator for a ghost particle in order not to violate the optical theorem, then the Feynman–Green function will propagate (forward in time) ghost particles with negative energy. This is tantamount to saying that ghosts possess negative energy, in agreement with the results of both classical and quantum mechanics. However, in this case, it is possible that the vacuum is not stable; for example out of vacuum it could be possible to create ghosts and normal particles, as relativistic kinematics allows this process to happen. This case has been discussed in detail in [12, 13], and we refer the reader to these works. Because the vacuum is unstable, the result is that ghost theories need a cut-off (which explicitly breaks Lorentz invariance) of order of a few MeV.

If it is clear that an imaginary speed of propagation for a particle gives a classical instability, for superluminal modes, things are not as easy. In fact, there have been some discussions if such modes might create problems of causality violations [14] and inconsistencies in the theory of black-hole thermodynamics [15, 16]. However, it is actually in the models studied in this paper, where it is clear that they in general lead to an ill-defined Cauchy problem. In fact, the speed of these modes not only is greater than 1, but actually in most of the cases it blows up at some instant of time at which no metric singularity appears. In other words, the dynamics of the background does not prevent these metric-perturbation modes from reaching an infinite speed. Therefore, if a mode reaches a positive-infinite squared-speed then, at that instant, the spatial section becomes in general causally connected. This gives rise to an ill-posed Cauchy problem and, in the end, to a classical instability as the squared-speed, immediately afterwards, acquires infinite negative values.

In this paper, we have reviewed the state of the art for some models of modified gravity and we have argued these models suffer from severe theoretical constraints, as they predict the existence of ghost-like, superluminal, and classically unstable modes on cosmological backgrounds.

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References

[1] Wetterich C 1988 *Nucl. Phys.* B **302** 668

[2] Steinhardt P J, Wang L M and Zlatev I 1999 *Phys. Rev.* D **59** 123504 (*Preprint* astro-ph/9812313)

[3] Dvali G R, Gabadadze G and Porrati M 2000 *Phys. Lett.* B **485** 208 (*Preprint* hep-th/0005016)

[4] Carroll S M, Duvvuri V, Trodden M and Turner M S 2004 *Phys. Rev.* D **70** 043528 (*Preprint* astro-ph/0306438)

[5] Wands D 1994 *Class. Quantum Grav.* **11** 269 (*Preprint* gr-qc/9307034)

[6] Carroll S M, De Felice A, Duvvuri V, Easson D A, Trodden M and Turner M S 2005 *Phys. Rev.* D **71** 063513 (*Preprint* astro-ph/0410031)

[7] Calcagni G, de Carlos B and De Felice A 2006 *Nucl. Phys.* B **752** 404 (*Preprint* hep-th/0604201)

[8] Nunez A and Solganik S 2005 *Phys. Lett.* B **608** 189 (*Preprint* hep-th/0411102)

[9] Chiba T 2005 *J. Cosmol. Astropart. Phys.* JCAP03(2005)008 (*Preprint* gr-qc/0502070)

[10] Navarro I and Van Acoleyen K 2006 Consistent long distance modification of gravity from inverse powers of the curvature *J. Cosmol. Astropart. Phys.* JCAP03(2006)008 (*Preprint* gr-qc/0511045)

[11] De Felice A, Hindmarsh M and Trodden M 2006 Ghosts, instabilities, and superluminal propagation in modified gravity models *J. Cosmol. Astropart. Phys.* JCAP08(2006)005 (*Preprint* astro-ph/0604154)

[12] Carroll S M, Hoffman M and Trodden M 2003 *Phys. Rev.* D **68** 023509 (*Preprint* astro-ph/0301273)

[13] Cline J M, Jeon S and Moore G D 2004 *Phys. Rev.* D **70** 043543 (*Preprint* hep-ph/0311312)

[14] Adams A, Arkani-Hamed N, Dubovsky S, Nicolis A and Rattazzi R 2006 *Preprint* hep-th/0602178

[15] Dubovsky S L and Sibiryakov S M 2006 Spontaneous breaking of Lorentz invariance, black holes and perpetuum *Phys. Lett.* B **638** 509 (*Preprint* hep-th/0603158)

[16] Babichev E, Mukhanov V F and Vikman A 2006 *J. High Energy Phys.* JHEP09(2006)061 (*Preprint* hep-th/0604075)